Math 255B Lecture 11 Notes

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1 The Cayley Transform of Symmetric Operators

1.1 The Cayley transform

Let $S : D(S) \to H$ be closed, symmetric, and densely defined. We have shown that $S \pm i$ are injective iff $\operatorname{Im}(S \pm i)$ is closed, and $\|(S+i)x\|^2 = \|(S-i)x\|^2$ $(= \|Sx\|^2 + \|x\|^2)$.

Definition 1.1. The Cayley transform T of S is the operator $T = (S+i)(S-i)^{-1} : \operatorname{Im}(S-i) \to \operatorname{Im}(S+i)$.

The above norm calculation shows that T is an isometric bijection.

Proposition 1.1. *$T - 1$ is injective, $\operatorname{Im}(T-1) = D(S)$, and $S = i(T+1)(T-1)^{-1} : D(S) \to H$.*

Proof. If $y \in \operatorname{Im}(S-i)$ with $y = (S-i)x$, then

$$(T-1)y = (S+i)x - (S-i)x = 2ix.$$

We get $T-1$ is injective and $\operatorname{Im}(T-1) = D(S)$. Similarly,

$$(T+1)y = (S+i)x + (S-i)x = 2Sx,$$

so

$$2S\frac{1}{2i}(T-1)y = (T+1)y.$$

Then

$$S = i(T+1)(T-1)^{-1}. \qquad \square$$

Conversely, let $H_1, H_2 \subseteq H$ be closed subspaces, and let $T : H_1 \to H_2$ be a unitary map be such that $\operatorname{Im}(T-1)$ is dense in H. We claim that $T-1$ is injective: If $(T-1)y = 0$ for $y \in H_1$, then for $z \in H_1$,

$$\langle y, (T-1)z\rangle = \langle y, Tz\rangle - \langle y, z\rangle = \langle Ty, Tz\rangle - \langle y, z\rangle = 0.$$

Define $S : D(S) = \mathrm{Im}(T-1) \to H$ by $S = i(T+1)(T-1)^{-1}$. We claim that S is symmetric. For $x = (T-1)y \in D(S)$,

$$\begin{aligned}\langle Sx, x\rangle &= i\,\langle (T+1)y, (T-1)y\rangle \\ &= i(\|Ty\|^2 - \langle Ty, y\rangle + \langle y, Ty\rangle - \|y\|^2) \\ &= i(-\langle Ty, y\rangle + \langle y, Ty\rangle) \in \mathbb{R}.\end{aligned}$$

We get $\langle Sx, x\rangle = \langle x, Sx\rangle$ for all $x \in D(S)$. Polarize this identity (i.e. $x = y + z, y + iz$) to get that S is symmetric.

We claim that S is closed. If $(x, z) \in \overline{G(S)}$, there is a sequence $y_n \in H_1$ such that $(T-1)y_n \to x$ and $i(T+1)y_n \to z$. So $y_n \to y \in H_1$. $Ty_n \to Ty$, so $(T-1)y_n \to (T-1)y = x \in D(S)$. Then

$$i(T+1)y_n \to i(T+1)y = i(T+1)(T-1)^{-1}x = Sx = z.$$

Finally, let T_1 be the Cayley transform of S, $T_1 : \mathrm{Im}(S-i) \to \mathrm{Im}(S+i)$ with $T_1 = (S+i)(S-i)^{-1}$. If $y \in D(S) = \mathrm{Im}(T-1)$ with $y = (T-1)x$ ($x \in H_1$), then

$$(S-i)y = (S-i)(T-1)x = i(T+1)x - i(T-1)x = 2ix.$$

So $D(T_1) = H_1 = D(T)$. Now we check

$$T_1 x = \frac{1}{2i} T_1 (S-i)y = \frac{1}{2i}(S+i)y = \frac{1}{2i}(\underbrace{S(T-1)x}_{i(T+1)x} + i(T-1)x) = Tx.$$

So the Cayley transform of S is T.

We summarize the results in a proposition.

Proposition 1.2. *Let S be closed, symmetric, and densely defined. Then the Cayley transform $T : \mathrm{Im}(S-i) \to \mathrm{Im}(S+i)$ sending $(S-i)x \mapsto (S+i)x$ is unitary,* $\mathrm{Im}(T-1) = D(S)$*, $T-1$ is injective, and $S = i(T+1)(T-1)^{-1}$. Conversely, if H_1, H_2 are closed subspaces of H, $T : H_1 \to H_2$ is unitary, and* $\mathrm{Im}(T-1)$ *is dense, then T is the Cayley transform of a unique symmetric, closed, densely defined operator S.*

1.2 Deficiency subspaces

Now we are ready to check whether a closed, symmetric, and densely defined operator is self-adjoint.

Definition 1.2. Let S be closed, symmetric, and densely defined. The **deficiency subspaces** associated to S are $D_\pm := (\mathrm{Im}(S \pm i))^\perp = \ker(S^* \mp i)$. The **deficiency indices** are $n_\pm = \dim D_\pm$ (Hilbert space dimension).

The deficiency indices measure the extent to which S may fail to be self-adjoint.
We know $D_\pm \subseteq D(S)$. Introduce also

$$D(S^*)|_{D_\pm} =: \widehat{D}_\pm = \{(x, S^*x) : x \in D_\pm\} = \{\langle x, \pm ix\rangle : x \in D_\pm\}.$$

Theorem 1.1. *Let S be closed, symmetric, and densely defined. Then*

$$G(S^*) = G(S) \oplus \widehat{D}_+ \oplus \widehat{D}_-,$$

where the direct sum is orthogonal.

Proof. Check first that $G(S) \perp \widehat{D}_\pm$ (we check $+$): if $x \in D(S)$ and $y_+ \in D_+$

$$\langle (x, Sx), (y_+, iy_+)\rangle = \langle x, y_+\rangle + \langle Sx, iy_+\rangle = \langle x, y_+\rangle + \underbrace{\langle x, iS^*y_+\rangle}_{=\langle x, iiy_+\rangle} = 0.$$

Also, $\widehat{D}_+ \perp \widehat{D}_-$:

$$\langle (y_+, iy_+), (y_-, -iy_-)\rangle = \langle y_+, y_-\rangle + \langle iy_+, (1/i)y_-\rangle = 0.$$

By orthogonality, $G(S) \oplus \widehat{D}_+ \oplus \widehat{D}_-$ is a closed subspace of $G(S^*)$. It remains to show that if (y, S^*y) is orthogonal to $G(S), \widehat{D}_\pm$, then $y = 0$. We will show this next time. □